Section 8: Derivatives and Rates of Change

Learning Objectives

- Identify the derivative of a function as the limit of a difference quotient.
- Interpret the meaning of a derivative within a problem.
- Solve problems involving the slope of a tangent line.

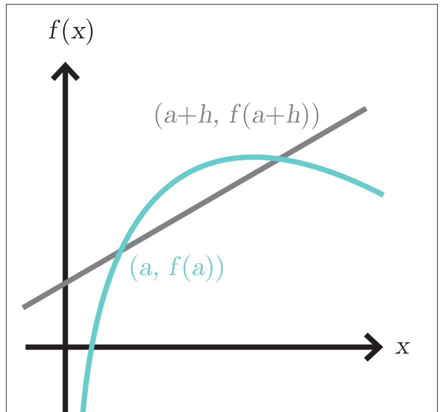
Two fundamental questions in calculus are answered with the same special type of limit. How do you find the tangent line to a curve at a given point? What is the velocity of a moving object at a particular instant? Both can be examined using the special limit called the derivative.

Tangents

You previously explored how to find the equations for a secant line to the graph of a function $f(x)$ for $x=a$. For a number $h>0$, a secant line passes through the graph of $f(x)$ at points $(a, f(a))$ and $(a+h,\ f(a+h))$.

If h is small, the slope of the secant line approximates the rate at which the function $f(x)$ is changing between the values a and $a+h$. Taking the limit of these slopes gives us the notion for the slope of the tangent line of $f(x)$ at $x=a$.

A secant line



The slope of the secant line is $\dfrac{f(a+h)-f(a)}{h}$

Definition

The tangent line to the graph of $y=f(x)$ at $x=a$ is the line that passes through $(a, f(a))$ with slope $m=\lim\limits_{h\to0}\dfrac{f(a+h)-f(a)}{h}$ when the limit exists.

Example 1: Finding a Tangent Line

Find an equation of the tangent line to the graph of $f(x)=3x^2$ for $a=1$.

Use $a=1$ in the limit to find the slope of the tangent line.

$m=\lim\limits_{h\to0}\dfrac{f(1+h)-f(1)}{h}=\lim\limits_{h\to0}\dfrac{3(1+h)^2-3(1)^2}{h}$	Use $a=1$ and $(x)=3x^2$.
$=\lim\limits_{h\to0}\dfrac{3(1^2+2h+h^2)-12}{h}=\lim\limits_{h\to0}\dfrac{(3+6h+3h)-3}{h}$	Expand and simplify.
$=\lim\limits_{h\to0}\dfrac{6h+3h^2}{h}=\lim\limits_{h\to0} 6+3h$	Cancel like terms.
$=6$	Evaluate the limit.

The slope of the tangent line is 6. As $f(1)=3(1)^2=3$, we know that the tangent line passes through the point $(1,\ 3)$. Therefore, an equation of the tangent line can be found using the point-slope form of the equation of a line.

$$y-3=6(x-1)$$
$$y=6x-3$$

Prerequisite Knowledge

- Evaluating limits using limit laws
- Using techniques to evaluate limits that indeterminate forms
- Finding the equation of a secant line for the graph of a function

Key Knowledge

Students will learn how to calculate the derivative of a function at a point by using the definition.

- Stress to students that the definition of the derivative comes directly from the slope formula.
- Make the connection between the numeric and graphical meaning of the derivative, analogous to that of the slope of a line.
- Point out that all limits that define derivatives result in the indeterminate form, so previously learned techniques are required.
- Incorporate different function types that will require varied algebraic techniques to evaluate the derivative. For instance, root functions require rationalizing the numerator, while rational functions require combining fractions.

Common Errors

$f(a+h)-f(a)=f(h)$ Remind students that in the definition of the slope of the tangent line, they must first evaluate $f(a+h)$ and $f(a)$. The resulting expression may then be simplified.

Open Question

What should you do when the limit results in $\frac{0}{0}$?

Additional Example

Find an equation of the line tangent to the graph of $f(x)=-2x^2$ for $a=3$.

Solution: $m=\lim\limits_{h\to0}\dfrac{-2(3+h)^2-(-2(3)^2)}{h}$

$=-12$

An equation of the tangent line is $y+18=-12(x-3)$, or $y=-12x+18$.